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The propagator for quantum mechanics on a group manifold from an infinite-dimensional analogue of the Duistermaat–Heckman integration formula

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Abstract. An exact expression for the quantum mechanical propagator for a particle moving on a group manifold is shown to arise from the application of an infinite-dimensional version of the Duistermaat-Heckman integration formula to a suitable path integral over the based loop group of the group. In an appendix the equivalence of this expression to a spectral representation of the propagator is demonstrated by means of a Poisson resummation.

1. Introduction

The quantum mechanics of a particle moving without external forces on a compact Lie group (quantum mechanics on a group for short) is an interesting model for testing approaches to quantisation in non-linear systems. It is a special case of the model of a quantum mechanical particle moving on a general Riemannian manifold, which has been studied both from the path integral point of view (Marinov 1980, Schulman 1981 ch 24, Grosche and Steiner 1987) and from the canonical point of view (de Witt 1952, 1957). The principle of preserving the symmetries of the classical system in the quantum Hamiltonian leads to an unambiguous canonical quantisation for the particle on a group manifold (Omote and Sato 1972, Charap 1973). In the path integral approach to this case a long-established feature is the exactness of the semiclassical approximation for the propagator (Schulman 1968, Dowker 1970, 1971, Marinov and Terentiev 1979).

Semiclassical exactness in a rather different sense is the hallmark of the Duistermaat-Heckman (DH) integration formula (Duistermaat and Heckman 1982, 1983, Berline and Vergne 1983). This is an integration formula for certain integrals on finite-dimensional compact symplectic manifolds. The integrals are of the form $\int \exp(H)\omega^n/n!$ where H is a function and ω is the sympletic form. The answer is given in terms of a sum over the critical points of H, and is identical to the answer which would be obtained by applying the 'semiclassical approximation' at each critical point (i.e. through evaluating the fluctuation determinant of H, whilst ignoring the fact that the critical point need not be a local maximum). In fact a better way of viewing the DH formula is in terms of localisation: the integrand can be shown to be exact

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everywhere outside the critical points of H, so that by excising infinitesimally small balls centred around the critical points and using Stokes' theorem, one sees that the answer can only depend on local data at the critical points of H. (For more information and examples of the DH formula see Picken (1988).)

It has been suggested by Atiyah that a generalisation of the DH formula to infinite dimensions would be a desirable goal (Atiyah 1985). Such a generalisation seems to give correct results in certain circumstances (e.g. when applied to the loop space of a manifold the index theorem is obtained (Atiyah 1985, Bismut 1985)). There are strong links to the localisation arguments used in the 'physicists proof' of the index theorem (Alvarez-Gaumé 1983, Friedan and Windey 1984, Getzler 1983) and in Witten's derivation of the Morse inequalities and other results via supersymmetric sigma models (Witten 1982). Quantum mechanics on a group would seem to be another fruitful area for testing such an infinite-dimensional version of the DH formula, as the Hilbert space of the theory, which is essentially the space, ΩG, of based loops on G, has been studied in depth by Pressley and Segal (1987), who have, in particular, elucidated the geometrical features of ΩG as an infinite-dimensional symplectic manifold.

In this paper it is shown that the propagator for quantum mechanics on a group manifold is correctly reproduced as the result of applying an infinite-dimensional. version of the DH integration formula to a suitable path integral over ΩG. In § 2 the model is presented and a spectral representation of the propagator is written down. This representation is equivalent to a sum-over-classical-paths representation (sum-over-functional-critical-points representation), as is shown in an appendix. In § 3 a path integral for the propagator is proposed and the necessary geometric features of ΩG are introduced. By choosing a suitable measure and applying an infinite-dimensional version of the DH formula the sum-over-classical-paths representation of the propagator is reproduced. In § 4 the conclusions are presented.

2. The propagator for quantum mechanics on G

The model we will be studying is the quantum mechanics of a point particle constrained to move on a group manifold, G, without external forces. The action governing the classical motion of the particle is

$$I[f] = -\tfrac{1}{2}\int_{t=t_0}^{t=t_1} \langle f^{-1}\dot{f}, f^{-1}\dot{f}\rangle \, \mathrm{d}t \tag{2.1}$$

where $f\colon [t_0, t_1] \to G$ gives the position of the particle as a function of time t. In terms of local coordinates $x = \{x^i, i = 1, \ldots, n = \dim G\}$ on the group manifold the action takes the form

$$I[x] = \tfrac{1}{2}\int_{t=t_0}^{t=t_1} g_{ij}(x)\dot{x}^i\dot{x}^j \, \mathrm{d}t \tag{2.2}$$

where $-g_{ij}(x)$ are the components of the (negative-definite) Cartan–Killing metric on G. Thus (2.2) is the constrained kinetic energy. The corresponding Hamiltonian function is

$$H = \tfrac{1}{2}g^{ij}(x)p_ip_j \tag{2.3}$$

where

$$p_i = g_{ij}(x)\dot{x}^j. \tag{2.4}$$

Following de Witt (1952) the Hamiltonian (2.3) is quantised by the replacement $p_i \to \hat{p}_i$, where $\hat{p}_i = -\mathrm{i}\hbar D_i$ and where D_i is the covariant derivative with respect to the Christoffel connection of the metric g (see also the discussion in Charap (1973)). Thus the Hamiltonian operator is

$$H = -\tfrac{1}{2}\hbar^2\Delta \tag{2.5}$$

where Δ is the Laplace-Beltrami operator on the group manifold:

$$\Delta = g^{ij}D_iD_j. \tag{2.6}$$

The propagator from an initial position g_0 at $t=0$ to a final position g_1 at $t=t_1$:

$$\mathcal{K}(g_0, g_1; t_1) = \langle g_1|\exp(-it_1H/\hbar)|g_0\rangle \tag{2.7}$$

has a spectral representation

$$\mathcal{K}(g_0, g_1; t_1) = \sum_n \psi_n(g_1)\bar{\psi}_n(g_0)\exp(-\mathrm{i}t_1E_n/\hbar) \tag{2.8}$$

in terms of orthonormal eigenfunctions $\psi_n(g)$ and eigenvalues E_n of H. The spectrum of H may be described as follows: for each irreducible representation $V(\lambda)$ of G, labelled by its highest weight λ, there are $\mathrm{d}(\lambda)^2$ independent eigenfunctions, where $\mathrm{d}(\lambda)$ is the dimension of $V(\lambda)$, with eigenvalue $-\frac{1}{2}\hbar^2c_2(\lambda)$, where $c_2(\lambda)$ is the eigenvalue of the second-order Casimir element, c_2, in the representation $V(\lambda)$. The eigenfunctions are the $\mathrm{d}(\lambda)^2$ entries of the matrix $\mathrm{d}(\lambda)^{1/2}D^\lambda(g)$ where $D^\lambda(g)$ is the matrix representing g in the representation $V(\lambda)$. Thus

$$\begin{aligned}\mathcal{K}(g_0, g_1; t_1) &= \sum_\lambda \mathrm{d}(\lambda)\,\mathrm{Tr}[D^\lambda(g_1)D^{\lambda^+}(g_0)]\exp(-\mathrm{i}\hbar t_1c_2(\lambda)/2)\\ &= \sum_\lambda \mathrm{d}(\lambda)\chi_\lambda(\varphi)\exp(-\mathrm{i}\hbar t_1c_2(\lambda)/2)\end{aligned} \tag{2.9}$$

where $g_1(g_0)^{-1}$ is conjugate to $\exp(\mathrm{i}\varphi)$, an element of the torus, T, of G. (As this does not determine φ uniquely we define φ to be the shortest element of it such that $g_1(g_0)^{-1}$ is conjugate to $\exp(\mathrm{i}\varphi)$. Here and elsewhere we make the assumption that g_1 and g_0 are not equal, nor are they conjugate points in the geometric sense.) Furthermore χ_λ is the character of the representation $V(\lambda)$:

$$\chi_\lambda(\varphi) = \mathrm{Tr}(D^\lambda(\mathrm{e}^{\mathrm{i}\varphi})). \tag{2.10}$$

Thus without loss of generality we may set $g_0 = 1$ and $g_1 = \exp(\mathrm{i}\varphi)$ and we do so henceforth. For further details of the derivation see Marinov and Terentiev (1979).

For the present purpose it is preferable to work with a different representation of the propagator, namely as a sum over a certain lattice $\check{T}$. $\check{T}$ is contained in it, where t is the torus Lie algebra, and is generated over the integers by the coroots $\{\tilde{\alpha}_i = 2\alpha_i/\langle\alpha_i, \alpha_i\rangle\}$, $i = 1, \ldots, l = \mathrm{rank}\, G$, where α_i, $i = 1, \ldots, l$ are the simple roots of G, identified with elements of it via the Cartan-Killing form $\langle\ ,\ \rangle$. Any $\nu \in \check{T}$ satisfies $\exp(2\pi\mathrm{i}\nu) = 1$ and there are no other elements of it with this property. Thus there is a one-to-one correspondence between $\check{T}$ and the set of closed geodesics in the torus of unit parameter length, starting and ending at 1 (ν corresponds to the geodesic $t \mapsto \exp(2\pi\mathrm{i}\nu t)$, $t \in [0, 1]$). The propagator $\mathcal{K}$, regarded as a function $K(\varphi, t_1)$ of φ and t_1, is expressed in this representation as

$$\begin{aligned}K(\varphi, t_1) &= c(2\pi\mathrm{i}\hbar t_1)^{-n/2}\exp(\mathrm{i}\hbar n t_1/48)\\ &\quad\times\left[\sum_{\nu\in\check{T}}\left(\prod_{\alpha\in R_+}\frac{\langle\alpha, \varphi+2\pi\nu\rangle}{2\sin(\langle\alpha, \varphi+2\pi\nu\rangle/2)}\right)\exp(\mathrm{i}\langle\varphi+2\pi\nu, \varphi+2\pi\nu\rangle/2\hbar t_1)\right]\end{aligned} \tag{2.11}$$

where $n = \dim G$, R_+ is the set of positive roots with respect to some chosen ordering of roots and c is a constant. Thus we recognise $K(\varphi, t_1)$ as a 'sum over classical paths', as the action (2.1) may be rewritten

$$I[f] = -\frac{1}{2}\frac{1}{t_1}\int_{t=0}^{1} \langle f^{-1}\dot{f}, f^{-1}\dot{f}\rangle \, dt \tag{2.12}$$

by rescaling the time. For the geodesic $f_\nu(t) = \exp[\mathrm{i}t(\varphi + 2\pi\nu)]$ this gives

$$I[f_\nu] = \langle \varphi + 2\pi\nu, \varphi + 2\pi\nu\rangle / 2t_1 . \tag{2.13}$$

The expression (2.11) was obtained by Marinov and Terentiev (1979) using the semiclassical approximation and they subsequently showed that it is an exact expression for the propagator by the indirect method of demonstrating that it satisfies the Schrödinger equation. In the appendix we give a direct derivation of the equality of the 'sum-over-classical-paths' representation (2.11) and the form of the propagator from the spectral representation (2.9). The derivation uses a Poisson resummation and various results from Lie algebra structure theory and representation theory. A similar calculation for the groups SU(n) was performed by Dowker (1971).

3. The propagator on G as an exact path integral

There is a slightly different way of viewing the classical actions (2.13). Starting with the time-rescaled action (2.12) one may utilise the fact that the manifold on which the particle moves is a group to factorise the path $f(t)$ (where $f(0) = 1$ and $f(1) = \exp(\mathrm{i}\varphi)$) into a closed loop $g(t)$ and a reference path $f_0(t)$:

$$f(t) = f_0(t)g(t). \tag{3.1}$$

Here the multiplication of paths is pointwise group multiplication of the image points and $f_0(t)$ is the shortest geodesic connecting the endpoints

$$f_0(t) = \exp(\mathrm{i}t\varphi). \tag{3.2}$$

In terms of $g(t)$ the functional I becomes

$$I[g] = -\frac{1}{2}\frac{1}{t_1}\int_{t=0}^{1} \langle g^{-1}\dot{g} + g^{-1}\mathrm{i}\varphi g, g^{-1}\dot{g} + g^{-1}\mathrm{i}\varphi g\rangle \, dt. \tag{3.3}$$

The classical paths (i.e. stationary points) for $I[g]$ are now the closed geodesics

$$g_\nu(t) = \exp(2\pi\mathrm{i}\nu t) \tag{3.4}$$

and one has

$$I[g_\nu] = \langle \varphi + 2\pi\nu, \varphi + 2\pi\nu\rangle / 2t_1 . \tag{3.5}$$

The advantage of the change of functional variable is that g is an element of ΩG, the space of based loops on G (i.e. loops starting and ending at $1 \in G$) and this space has some very special geometric features, which may be summarised by saying that ΩG is an infinite-dimensional flag manifold (see Pressley and Segal (1987) or Freed (1985); an expository account of finite-dimensional flag manifolds and the Duistermaat-Heckman integration formula is given in Picken (1988)). For the present purpose the key feature of ΩG is that it is a Kähler manifold and thus in particular it possesses a symplectic form ω. Now on finite-dimensional symplectic manifolds the Duistermaat–Heckman (DH) integration formula (to be described below) asserts that a certain class of integrals, of the type $\int \exp(\mathrm{i}H)\,\mathrm{d}\mu$, localises to the critical points of the function H, i.e. the answer depends only on local data at the critical points of H. Our purpose is to show that with an appropriate choice of measure $\mathrm{d}\mu[g]$ the integral

$$\int_{\Omega G} \exp(\mathrm{i}I[g]/\hbar)\,\mathrm{d}\mu[g] \tag{3.6}$$

belongs to the same class and reproduces the exact propagator $K(\varphi, t_1)$, under the assumption that the DH result extends to infinite dimensions.

First we describe a version of the DH result for finite-dimensional symplectic manifolds. Let (M, ω) be a compact symplectic manifold of dimension $2n$ and H be a smooth function (the Hamiltonian) on M. The corresponding Hamiltonian vector field, ξ, is given by

$$\mathrm{d}H + \iota_\xi \omega = 0. \tag{3.7}$$

Assume furthermore that H has only isolated critical points $\{m_i\}_{i\in I}$ on M (this is the simplest case). Then the DH formula asserts

$$\int_M \exp(\mathrm{i}H)(\omega/2\pi\mathrm{i})^n/n! = \sum_{j=I} \exp(\mathrm{i}H(m_j))/\mathrm{Pf}(J_\xi(m_j)). \tag{3.8}$$

On the right-hand-side $\mathrm{Pf}(J_\xi(m_i))$ is a kind of winding number of the vector field ξ around its zero at m_i. Specifically J_ξ is a linear automorphism of $T_{m_i}M$ defined by

$$J_\xi(Y) = -\mathscr{L}_\xi(Y) \qquad \forall\, Y \in T_{m_i}M \tag{3.9}$$

where $\mathscr{L}_\xi$ is the Lie derivative. $\mathrm{Pf}(J_\xi(m_i))$ is then the Pfaffian of the antisymmetric map J_ξ.

Following Atiyah and Bott (1984) we introduce local coordinates $\{x_j, y_j\}, j = 1, \ldots, n$ around the point m_i, which vanish at m_i, in which H, ω and ξ take the following form:

$$H = H(m_i) + \sum_{j=1}^{n} \lambda_j\mu_j(x_j^2 + y_j^2)/2 + \ldots \tag{3.10}$$

$$\omega = \sum_{j=1}^{n} \mu_j\,\mathrm{d}x_j \wedge \mathrm{d}y_j + \ldots \tag{3.11}$$

$$\xi = \sum_{j=1}^{n} \lambda_j(x_j\partial_{y_j} - y_j\partial_{x_j}) + \ldots \tag{3.12}$$

where λ_j, μ_j are constants and the dots indicate terms with coefficients of higher polynomial order in the coordinates. On the subspace of $T_{m_i}M$ spanned by $\{\partial_{x_j}, \partial_{y_j}\}$, J_ξ acts as the matrix $\left(\begin{smallmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{smallmatrix}\right)$. Defining the Pfaffian of J_ξ to be the product of the upper right-hand entries of these matrices, one has

$$\mathrm{Pf}(J_\xi(m_i)) = \prod_{j=1}^{n} \lambda_j. \tag{3.13}$$

However, using (3.10) and (3.11), one can give an alternative expression for $\mathrm{Pf}(J_\xi)$, namely

$$\mathrm{Pf}(J_\xi(m_i)) = \frac{(\det(\mathrm{Hess}\, H(m_i)))^{1/2}}{|\omega|(m_i)} \tag{3.14}$$

where the determinant of the Hessian of H is

$$\det(\text{Hess } H(m_i)) = \left(\prod_{j=1}^{n} \lambda_j \mu_j\right)^2 \tag{3.15}$$

and the determinant of ω at m_i, $|\omega|(m_i)$, is defined to be

$$|\omega|(m_i) = \prod_{j=1}^{n} \mu_j. \tag{3.16}$$

This alternative form for $\text{Pf}(J_\xi)$ turns out to be convenient for studying the infinite-dimensional generalisation.

Next we turn to the symplectic geometry of the space of based loops ΩG. One way of regarding ΩG (Pressley and Segal 1987, Freed 1985) is as a homogeneous space $\mathscr{G}/C(\mathscr{T}_0)$ where $\mathscr{G} = \mathbb{T} \ltimes LG$ (the semidirect product of the free loops on G, LG, by the circle action, $\mathbb{T}$, which rotates the loops) and $C(\mathscr{T}_0)$ is $\mathbb{T} \times G$ where G stands for the constant loops. $C(\mathscr{T}_0)$ is the centraliser of the subtorus $\mathscr{T}_0 = \mathbb{T} \times 1$ of the full torus $\mathscr{T} = \mathbb{T} \times T$ of $\mathscr{G}$. In finite dimensions homogeneous spaces of the form G/T or $G/C(T_0)$ are known as flag manifolds. Thus ΩG is an infinite-dimensional flag manifold. In finite dimensions flag manifolds possess left-G-invariant symplectic forms. The corresponding left-$\mathscr{G}$-invariant sympletic form, ω, on ΩG is described as follows (Pressley 1982). The tangent space to ΩG at the identity, e, may be identified with $L\mathfrak{g}/\mathfrak{g}$, where $L\mathfrak{g}$ is the Lie algebra of free loops on $\mathfrak{g}$ and $\mathfrak{g}$ are the constant loops. Thus $X \in T_e\Omega G$ corresponds to $X = \Sigma_{n \neq 0} X_n \exp(2\pi i n t)$ with $X_n \in \mathfrak{g}_c$. Then ω_e, regarded as a skew-bilinear form on $T_e\Omega G$, is given by

$$\omega_e(X, Y) = (-1/\hbar t_1) \int_{t=0}^{1} \langle \dot{X}, Y \rangle \, dt \tag{3.17}$$

where we have chosen the normalisation to suit our later purposes. Through left-$\mathscr{G}$-invariance, ω is then determined on all of ΩG.

The precise formulation of our result can now be stated.

If (a) in the path integral (3.6) one chooses the measure to be

$$d\mu[g] = Cc(2\pi i \hbar t_1)^{-n/2} \exp(i\hbar n t_1/48)(\text{Svol})_\infty[\omega/2\pi i] \tag{3.18}$$

where C is a regularisation factor and where $(\text{Svol})_\infty$ is the infinite-dimensional symplectic volume (see below)

and (b) one assumes that the DH formula may be applied on ΩG,

then the path integral reproduces the exact propagator (2.11).

In the above the symplectic volume is defined by analogy with the finite-dimensional case. Let

$$\omega = (1/2!) \sum_{\alpha,\beta} \omega_{\alpha\beta} \, e^\alpha \wedge e^\beta \tag{3.19}$$

be the expression for ω with respect to some basis of 1-forms $\{e^\alpha\}$, where the range of α, β is left unspecified (finite or infinite). Then in finite dimensions, $d = 2n$, one may write the symplectic volume as

$$(\text{Svol})_\delta[\omega] = \omega^n/n! = |\omega| \bigwedge_{\alpha=1}^{2n} e^\alpha \tag{3.20}$$

where $\bigwedge_{\alpha=1}^{2n} e^{\alpha}$ is the ordered wedge product with respect to some ordering of the basis, and the same ordering is applied in the calculation of $|\omega| = \det[\omega_{\alpha\beta}]$. By extension one defines the infinite-dimensional symplectic volume to be

$$(\mathrm{Svol})_{\infty}[\omega] = |\omega| \bigwedge_{\alpha=1}^{\infty} e^{\alpha}. \tag{3.21}$$

Following assumption (b) we now proceed to calculate the right-hand side of the DH formula (3.8) for the path integral (3.6) with measure (3.18). As above one identifies $T_e\Omega G$ with $L\mathfrak{g}/\mathfrak{g}$. In fact $X \in L\mathfrak{g}/\mathfrak{g}$ may be identified with a left-invariant vector field $\tilde{X}$ on ΩG. In the usual way these vector fields are dual to left-invariant 1-forms $g^{-1}\delta g$ via

$$(g^{-1}\delta g)(\tilde{X}_g) = X. \tag{3.22}$$

The first variation of I and the second variation of I when the first variation vanishes are

$$\delta I[g] = (-1/t_1)\int_{t=0}^{1}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(g^{-1}\delta g), g^{-1}\dot{g} + g^{-1}\mathrm{i}\varphi g\right\rangle dt \tag{3.23}$$

$$\delta^2 I[g] = (-1/t_1)\int_{t=0}^{1}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(g^{-1}\delta g), \left\{\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{ad}(g^{-1}\dot{g} + g^{-1}\mathrm{i}\varphi g)\right\} g^{-1}\delta g\right\rangle dt \tag{3.24}$$

so that at the geodesic $g = g_\nu$ (3.4)

$$\delta^2 I(\tilde{X}_{g_\nu}, \tilde{Y}_{g_\nu}) = (-1/t_1)\int_{t=0}^{1}\left\langle \dot{X}, \left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{i}\,\mathrm{ad}(\varphi + 2\pi\nu)\right) Y\right\rangle \mathrm{d}t. \tag{3.25}$$

In order to proceed it is convenient to define a basis for $L\mathfrak{g}/\mathfrak{g}$. Let

$$\mathfrak{g}_c = \mathfrak{h} + \sum_{\alpha\in R} \mathfrak{g}^{\alpha} \tag{3.26}$$

be the root space decomposition of $\mathfrak{g}_c$ and let $\{H_i\}$, $i = 1, \ldots, l$ be a basis for the Cartan subalgebra $\mathfrak{h} = \mathfrak{t}_c$, and E_α be a basis element for the root space $\mathfrak{g}^{\alpha}$. It is always possible to choose these bases such that

$$\begin{aligned}
&H_i^{\dagger} = -H_i && i = 1, \ldots, l \\
&E_\alpha^{\dagger} = E_{-\alpha} && \forall \alpha \in R \\
&\langle H_i, H_j\rangle = -\delta_{ij} && \\
&\langle H_i, E_\alpha\rangle = 0 && \\
&\langle E_\alpha, E_\beta\rangle = 1 && \beta = -\alpha \\
&\qquad\qquad\;\; = 0 && \text{otherwise.}
\end{aligned} \tag{3.27}$$

Now define the following basis elements of $L\mathfrak{g}\mathfrak{g}$:

$$\begin{aligned}
&V_j^n = (1/\sqrt{2})H_j[\exp(2\pi\mathrm{i}nt) + \exp(-2\pi\mathrm{i}nt)] && j = 1, \ldots, l;\ n > 0 \\
&W_j^n = (1/\sqrt{2})\mathrm{i}H_j[\exp(2\pi\mathrm{i}nt) - \exp(-2\pi\mathrm{i}nt)] && j = 1, \ldots, l;\ n > 0 \\
&V_\alpha^n = (1/\sqrt{2})(E_\alpha \exp(2\pi\mathrm{i}nt) - E_{-\alpha}\exp(-2\pi\mathrm{i}nt)) && \alpha \in R_+;\ n \neq 0 \\
&W_\alpha^n = (1/\sqrt{2})(\mathrm{i}E_\alpha \exp(2\pi\mathrm{i}nt) + iE_{-\alpha}\exp(-2\pi\mathrm{i}nt)) && \alpha \in R_+;\ n \neq 0.
\end{aligned} \tag{3.28}$$

By inserting this set in (3.25) one finds that it diagonalises the Hessian of I at g_ν. The eigenvalues of the Hessian (corresponding to the coefficients $\lambda_j\mu_j$ in (3.10)) are given by:

Basis element	Eigenvalue of Hess($I[g_\nu]$)
$V^n_j,\ W^n_j$	$(2\pi n)^2/t_1$
$V^n_\alpha,\ W^n_\alpha$	$(2\pi)^2 n(n+\langle\alpha,\varphi+2\pi\nu\rangle)/t_1$

Next one obtains the components of the symplectic form ω at g_ν using the same basis elements. From the expression (3.17) for the components of ω at e and from left-invariance one has

$$\begin{aligned}\omega_{g_\nu}(\tilde V^n_K, \tilde W^m_L) &= \omega_e(\tilde V^n_K, \tilde W^m_L)\\ &= (-1/\hbar t_1)\int_{t=0}^{1}\langle \dot V^n_K, W^m_L\rangle\,\mathrm{d}t\\ &= (2\pi n/\hbar t_1)\delta^{nm}\delta_{KL}\end{aligned} \tag{3.29}$$

where the indices K, L stand for j or α. Now with the aid of the regularisation factor

$$C=\left(\prod_{j=1}^{l}\left(\prod_{n>0}2\pi n\right)\right)\left(\prod_{\alpha\in R_+}\left(\prod_{n\neq 0}2\pi n\right)\right) \tag{3.30}$$

one obtains the regularised Pfaffian of J_ξ at g_ν:

$$\begin{aligned}\mathrm{Pf}(J_\xi(g_\nu))/C &= (\det(\mathrm{Hess}(I[g_\nu]/\hbar)))^{1/2}/C|\omega|(g_\nu)\\ &= \left[\prod_{j=1}^{l}\left(\prod_{n>0}[(2\pi n)^2/\hbar t_1)]\right)\right]\left[\prod_{\alpha\in R_+}\left(\prod_{n\neq 0}[(2\pi)^2 n(n+\langle\alpha,\varphi+2\pi\nu\rangle)/\hbar t_1]\right)\right]\\ &\quad\times\left\{C\left[\prod_{j=1}^{l}\left(\prod_{n>0}(2\pi n/\hbar t_1)\right)\right]\left[\prod_{\alpha\in R_+}\left(\prod_{n\neq 0}(2\pi n/\hbar t_1)\right)\right]\right\}^{-1}\\ &= \prod_{\alpha\in R_+}\left(\prod_{n\neq 0}\left(1+\frac{\langle\alpha,\varphi+2\pi\nu\rangle}{2\pi n}\right)\right)\\ &= \prod_{\alpha\in R_+}\left(\prod_{n=1}^{\infty}\left(1-\left(\frac{\langle\alpha,\varphi+2\pi\nu\rangle}{2\pi n}\right)^2\right)\right)\\ &= \prod_{\alpha\in R_+}\left(\frac{\sin(\langle\alpha,\varphi+2\pi\nu\rangle/2)}{\langle\alpha,\varphi+2\pi\nu\rangle/2}\right)\end{aligned} \tag{3.31}$$

(using a standard infinite product formula in the last equality). Combining this result with the expression for the measure (3.18), the DH formula yields

$$\begin{aligned}\int_{\Omega G}\exp(\mathrm{i}I[g]/\hbar)\,\mathrm{d}\mu[g] &= c(2\pi\mathrm{i}\hbar t_1)^{-n/2}\exp(\mathrm{i}\hbar t_1 n/48)\left[\sum_{\nu\in\hat T}\left(\prod_{\alpha\in R_+}\frac{\langle\alpha,\varphi+2\pi\nu\rangle}{2\sin(\langle\alpha,\varphi+2\pi\nu\rangle/2)}\right)\right.\\ &\quad\left.\times\exp(\mathrm{i}\langle\varphi+2\pi\nu,\varphi+2\pi\nu\rangle/2\hbar t_1)\right]\end{aligned} \tag{3.32}$$

in agreement with (2.11).

4. Conclusions

The result just shown is an encouraging piece of evidence that the DH formula does indeed extend to infinite dimensions in suitable circumstances. However, there remain several questions to be answered. Firstly, one would like to have a better understanding of the factors multiplying the symplectic volume in the expression (3.18) for the path integral measure. The regularisation factor C might be tolerated as it seems to be a necessary concomitant of working in infinite dimensions (cf Atiyah 1985), but the time-dependent factors had to be put in by hand. Secondly, a proof is still lacking, although one may be able to learn something from studying the DH formula on the finite-dimensional analogues of ΩG, i.e. on flag manifolds (Picken 1988). As a final remark it should be emphasised that this result might point the way to a new global approach to path integration which could avoid the ambiguities of the time-slicing definition of the path integral whilst retaining the spirit of Feynman's original idea.

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Appendix

We start with a discussion of the weight lattice $\hat{T}$ and its dual, the coweight lattice $\check{T}$. Weights, being elements of $\mathfrak{h}^*$, the dual of $\mathfrak{h}$, which is the complexification of the torus Lie algebra $\mathfrak{t}$, may be identified with elements of $\mathfrak{h}$ itself via the Cartan–Killing form. Thus a weight λ is identified with an element of $\mathfrak{h}$, also denoted λ, via

$$\lambda(h) = \langle \lambda, h \rangle \qquad \forall h \in \mathfrak{h}. \tag{A1}$$

The weights form a lattice $\hat{T}$ generated over the integers by the basic weights $\{\lambda_i\}$, $i = 1, \ldots, l = \operatorname{rank} G$. The lattice $\hat{T}$ is contained in $\mathrm{i}\mathfrak{t} \subset \mathfrak{h}$. The Cartan–Killing form, which is negative definite on $\mathfrak{t}$ is positive definite on $\mathrm{i}\mathfrak{t}$. Dual to $\hat{T}$ in $\mathrm{i}\mathfrak{t}$ is the coweight lattice $\check{T}$ generated over the integers by the coroots $\{\tilde{\alpha}_i = 2\alpha_i/\langle \alpha_i, \alpha_i \rangle\}$, $i = 1, \ldots, l$, where $\{\alpha_i\}$, $i = 1, \ldots, l$, are the simple roots. The duality is expressed by

$$\langle \lambda_i, \tilde{\alpha}_j \rangle = \delta_{ij}. \tag{A2}$$

The coroots $\tilde{\alpha}_i$ possess the property

$$\exp(2\pi \mathrm{i} \tilde{\alpha}_i) = 1 \tag{A3}$$

and are the only elements of $\mathfrak{h}$ with this property. Thus there is a one-to-one correspondence between $\check{T}$ and the group of homomorphisms from $\mathbb{T}$, the circle group, to T. This group is generated by the homomorphisms $\{\eta_{\alpha_j}\}$, $j = 1, \ldots, l$ defined by

$$\eta_{\alpha_j}(\mathrm{e}^{\mathrm{i}\theta}) = \exp(\mathrm{i}\theta\tilde{\alpha}_j) \tag{A4}$$

where $\theta \in [0, 2\pi)$ parametrises $\mathbb{T}$. Geometrically these homomorphisms are geodesics in T.

Furthermore we will be needing the $l \times l$ matrices D, D' defined by

$$D_{ij} = \langle \tilde{\alpha}_i, \tilde{\alpha}_j \rangle \tag{A5}$$

$$D'_{ij} = \langle \lambda_i, \lambda_j \rangle. \tag{A6}$$

Let the two bases in it be related by

$$\lambda_i = M_{ij} \tilde{\alpha}_j \tag{A7}$$

$$\tilde{\alpha}_i = M^{-1}_{ij} \lambda_j. \tag{A8}$$

Then, by taking the Cartan–Killing bracket of (A7) with $\tilde{\alpha}_k$, respectively of (A8) with λ_K, we establish $D = M^{-1}$, $D' = M$ and thus

$$D' = D^{-1}. \tag{A9}$$

The expression for the propagator $K(\varphi, t)$ as a 'sum over classical paths' is

$$K(\varphi, t) = c(2\pi \mathrm{i} \hbar t)^{-n/2} \exp(\mathrm{i} \hbar n t/48)$$
$$\times \left[\sum_{\nu \in \check{T}} \left(\prod_{\alpha \in R_+} \frac{\langle \alpha, \varphi + 2\pi\nu \rangle}{2 \sin(\langle \alpha, \varphi + 2\pi\nu \rangle/2)} \right) \right.$$
$$\left. \times \exp(\mathrm{i} \langle \varphi + 2\pi\nu, \varphi + 2\pi\nu \rangle / 2\hbar t_1) \right] \tag{A10}$$

where c is a constant, $n = \dim G$ and R_+ is the set of positive roots with respect to some ordering. The elliptic function $J(\varphi)$, for $\varphi \in$ it, is defined by

$$J(\varphi) = \prod_{\alpha \in R_+} 2\mathrm{i} \sin(\langle \alpha, \varphi \rangle / 2) \tag{A11}$$

which may be rewritten as

$$J(\varphi) = \sum_{w \in W} (-1)^{|w|} \exp(\mathrm{i} \langle w\rho, \varphi \rangle). \tag{A12}$$

Here W is the Weyl group (the discrete group of linear transformations of it generated by reflections in the hyperplanes perpendicular to the simple roots), $(-1)^{|w|}$ is the determinant of the Weyl group element w, and ρ is the special weight:

$$\rho = \tfrac{1}{2} \sum_{\alpha \in R_+} \alpha = \sum_{i=1}^{l} \lambda_i \tag{A13}$$

familiar in representation theory. As $\rho \in \hat{T}$, (A12) implies

$$J(\varphi + 2\pi\nu) = J(\varphi) \tag{A14}$$

and hence we may take a factor $(\mathrm{i})^p / J(\varphi)$ (where p = number of positive roots = $(n - l)/2$) out of the square brackets in (A10):

$$K(\varphi, t) = c(2\pi \mathrm{i} \hbar t)^{-n/2} \exp(\mathrm{i} \hbar n t/48) (\mathrm{i})^p J(\varphi)$$
$$\times \left[\sum_{\nu \in \check{T}} \left(\prod_{\alpha \in R_+} \langle \alpha, \varphi + 2\pi\nu \rangle \right) \exp(\mathrm{i} \langle \varphi + 2\pi\nu, \varphi + 2\pi\nu \rangle / 2\hbar t) \right]. \tag{A15}$$

The remaining term in square brackets is of the form

$$\sum_{\nu\in \check{T}} g(2\pi\nu) \tag{A16}$$

where

$$g(x)=f(\varphi+x) \tag{A17}$$

and

$$f(x)=\left(\prod_{\alpha\in R_-}\langle\alpha, x\rangle\right)\exp(\mathrm{i}\langle x, x\rangle/2\hbar t). \tag{A18}$$

Expanding x as $x=x_i\tilde{\alpha}_i$ and regarding g as a function of the components $\{x_i\}$, the Poisson resummation formula

$$\sum_{\nu_i\in\mathbb{Z}} g(2\pi\nu_1,\ldots,2\pi\nu_l)=(1/2\pi)^l\sum_{\mu_i\in\mathbb{Z}}\hat{g}(\mu_1,\ldots,\mu_l) \tag{A19}$$

may be applied. Here the Fourier transform $\hat{g}$ is defined by

$$\hat{g}(k_1,\ldots,k_l)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x_1,\ldots,x_l)\exp(-\mathrm{i}k_jx_j)\,\mathrm{d}x_1\ldots\mathrm{d}x_l. \tag{A20}$$

Now from (A17) one has

$$\hat{g}(k_1,\ldots,k_l)=\exp(\mathrm{i}k_j\varphi_j)\hat{f}(k_1,\ldots,k_l). \tag{A21}$$

The Fourier transform of f is found to be

$$\hat{f}=\frac{(2\pi\mathrm{i}\hbar t)^{l/2}}{(\det D)^{1/2}}\left(\prod_{\alpha\in R_+}((\alpha)_iD^{-1}_{ij}k_j\hbar t)\right)\exp(-\mathrm{i}k_iD^{-1}_{ij}k_j\hbar t/2). \tag{A22}$$

Here $(\alpha)_i$ are the components of the root α with respect to the basis $\{\lambda_i\}$.

Digressing briefly to derive this last formula one has, from (A18) and (A20),

$$\hat{f}(k_1,\ldots,k_l)=\left(\prod_{\alpha\in R_+}(\alpha)_i\frac{1}{-\mathrm{i}}\frac{\partial}{\partial k_i}\right)(\exp(\mathrm{i}\langle x, x\rangle/2\hbar t))^\wedge \tag{A23}$$

where

$$(\exp(\mathrm{i}\langle x, x\rangle/2\hbar t))^\wedge=\frac{(2\pi\mathrm{i}\hbar t)^{l/2}}{(\det D)^{1/2}}\exp(-\mathrm{i}k_iD^{-1}_{ij}k_j\hbar t/2). \tag{A24}$$

It is a remarkable feature of the situation at hand that the formula (A22) is the only contribution from the multiple derivatives in (A23). In general one would expect extra terms from derivatives hitting polynomial factors in the k_i pulled down by previous derivatives. These extra terms may be shown to vanish by the following argument (cf Marinov and Terentiev 1979). The covariant Laplacian, Δ, when restricted to functions on the torus only, takes the form:

$$\Delta f=J^{-1}(D^{-1})_{ij}\partial_{\varphi_i}\partial_{\varphi_j}(Jf)+\langle\rho,\rho\rangle f \tag{A25}$$

where $\varphi=\varphi_i\tilde{\alpha}_i$. The constant term $\langle\rho,\rho\rangle f$ in (A25) ensures that $\Delta 1=0$, as may be checked by inserting the expression (A12) for J in (A25) and using $\langle w\rho, w\rho\rangle = \langle\rho,\rho\rangle\ \forall w\in W$. On the other hand, inserting instead the expression (A11) for J, $\Delta 1=0$ leads to the identity

$$\sum_{\beta\neq\gamma\in R_+}\langle\beta,\gamma\rangle\cos(\langle\beta-\gamma,\varphi\rangle/2)\left(\prod_{\substack{\alpha\in R_+\\ \alpha\neq\beta,\gamma}}\sin(\langle\alpha,\varphi\rangle/2)\right)=0. \tag{A26}$$

Then scaling φ by a real factor s, multiplying (A26) by s^{p-2} and letting $s\to 0$, one deduces an identity for any $\varphi \in$ it

$$\sum_{\beta\neq\gamma\in R_+} \langle\beta,\gamma\rangle \Bigg(\prod_{\substack{\alpha\in R_+\\ \alpha\neq\beta,\gamma}} \langle\alpha,\varphi\rangle\Bigg) = 0. \tag{A27}$$

From (A27) one may derive further identities by applying the operator $D_{ij}^{-1}\partial_{\varphi_i}\partial_{\varphi_j}$, the next one being

$$\sum_{\substack{\beta,\gamma,\delta,\varepsilon\in R_+\\ \text{no two equal}}} \langle\beta,\gamma\rangle\langle\delta,\varepsilon\rangle \Bigg(\prod_{\substack{\alpha\in R_+\\ \alpha\neq\beta,\gamma,\delta,\varepsilon}} \langle\alpha,\varphi\rangle\Bigg) = 0. \tag{A28}$$

Now the extra terms which might arise from the derivatives in (A23) all vanish due to the identities (A27), (A28), etc.

Returning to the main discussion, (A15)-(A22) may be combined to give

$$K(\varphi,t) = \frac{c}{(2\pi)^{l+p}(\det D)^{1/2}} \exp(i\hbar nt/48)(1/J(\varphi)) \times \Bigg[\sum_{\lambda\in\hat{T}} \Bigg(\prod_{\alpha\in R_+} \langle\alpha,\lambda\rangle\Bigg) \exp(\mathrm{i}\langle\lambda,\varphi\rangle)\exp(-\mathrm{i}\hbar t\langle\lambda,\lambda\rangle/2)\Bigg] \tag{A29}$$

where $\lambda = k_i\lambda_i$ with $k_i\in\mathbb{Z}$ is a general element of the weight lattice $\hat{T}$.

Because of the factor $\Pi_{\alpha\in R_+}\langle\alpha,\lambda\rangle$ certain weights are excluded from the summation in (A29), namely those weights of the form $w\lambda$ where $w\in W$, $\lambda\in\hat{C}_0$, but $\lambda\notin\hat{C}_0+\rho$. Here $\hat{C}_0$ denotes the intersection of the weight lattice $\hat{T}$ and the positive Weyl chamber C_0. That this is so may be seen by firstly supposing $\langle\alpha,\lambda\rangle=0$ for some $\alpha\in R_+$. Then $\langle w\alpha, w\lambda\rangle = 0$ for any $w\in W$ and either $w\alpha$ or $-w\alpha$ is positive. As any λ can be brought into $\hat{C}_0$ by a Weyl transformation it remains to show that the factor (A30) vanishes for $\lambda\in\hat{C}_0$, $\lambda\notin\hat{C}_0+\rho$. Letting $\lambda = k_i\lambda_i$ with not all $k_i>0$ we suppose $k_j=0$. Then from (A2) $\langle\alpha_j,\lambda\rangle=0$ and the assertion is proved. This may then be used to resum the expression in square brackets in (A29), yielding

$$K(\varphi,t) = \frac{c}{(2\pi)^{l+p}(\det D)^{1/2}} \times \exp(\mathrm{i}\hbar nt/48)(1/J(\varphi)) \Bigg[\sum_{\lambda\in\hat{C}_0}\sum_{w\in W}\Bigg(\prod_{\alpha\in R_+}\langle\alpha, w(\lambda+\rho)\rangle\Bigg) \times \exp(\mathrm{i}\langle w(\lambda+\rho),\varphi\rangle)\exp(-\mathrm{i}\hbar t\langle\lambda+\rho,\lambda+\rho\rangle/2)\Bigg]. \tag{A30}$$

Each $\lambda\in\hat{C}_0$ may be regarded as the highest weight of an irreducible representation $V(\lambda)$ of G. Thus one may invoke various results from representation theory. In particular one has the Weyl character formula for $V(\lambda)$:

$$\chi_\lambda(\varphi) = (1/J(\varphi))\sum_{w\in W}(-1)^{|w|}\exp(\mathrm{i}\langle w(\lambda+\rho),\varphi\rangle) \tag{A31}$$

the Weyl dimension formula for $\mathrm{d}(\lambda) = \dim V(\lambda)$:

$$\delta(\lambda) = \Bigg(\prod_{\alpha\in R_+}\langle\alpha,\lambda+\rho\rangle\Bigg)\Bigg(\prod_{\alpha\in R_+}\langle\alpha,\rho\rangle\Bigg)^{-1} \tag{A32}$$

and the formula for the eigenvalue $c_2(\lambda)$ of the second-order Casimir element $c_2 = -g^{ij}X_iX_j$ (where $[g^{ij}]=[g_{ij}]^{-1}$, $g_{ij}=-\langle X_i, X_j\rangle$ and $\{X_i\}$, $i=1,\ldots,n$ is a basis of $\mathfrak{g}$) in the representation $V(\lambda)$:

$$c_2(\lambda)=\langle\lambda+\rho, \lambda+\rho\rangle-\langle\rho,\rho\rangle. \tag{A33}$$

Combining (A30)-(A33) with the observation

$$\prod_{\alpha\in R_+}\langle\alpha, w\lambda\rangle=\prod_{\alpha\in R_+}\langle w^{-1}\alpha,\lambda\rangle=(-1)^{|w|}\prod_{\alpha\in R_+}\langle\alpha,\lambda\rangle \tag{A34}$$

the value of the constant c given by Marinov and Terentiev (1979) (in their notation $c=C_R^{-1}/N_W$)

$$c=(2\pi)^{p+l}(\det D)^{1/2}\left(\prod_{\alpha\in R_+}\langle\alpha,\rho\rangle\right)^{-1} \tag{A35}$$

and the Freudenthal-de Vries 'strange formula'

$$\langle\rho,\rho\rangle=n/24 \tag{A36}$$

one finally arrives at

$$K(\varphi,t)=\sum_{\lambda\in\hat{C}_0}\mathrm{d}(\lambda)\chi_\lambda(\varphi)\exp(-\mathrm{i}\hbar tc_2(\lambda)/2) \tag{A37}$$

which is the form of the propagator derived from the spectral representation.

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